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1987 J. Phys. A: Math. Gen. 20 6581

(http://iopscience.iop.org/0305-4470/20/18/053)

J. Phys. A: Math. Gen. **20** (1987) 6581-6589. Printed in the UK

The Rarita–Schwinger paradoxes

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Received 16 December 1986, in final form 30 June 1987

Abstract. The Rarita-Schwinger field coupled to an external electromagnetic field is reduced to a constrained mechanical model. It is shown that, guided by the mechanical model with the linear supplementary condition, both the non-positive definiteness of the anticommutators and the non-causal modes of propagation (the Rarita-Schwinger paradoxes) have the same origin.

1. Introduction

High-spin field theories are beset with difficulties of various kinds. In 1961 Johnson and Sudarshan [1] first observed the non-positive definiteness of the anticommutators for the Rarita-Schwinger field coupled to an external electromagnetic field. This anomaly was re-emphasised by Velo and Zwanziger [2] in 1969 who discovered non-causal modes of propagation in the classical theory. Many analyses have been made of these problems [3-5]. We now have recognised that these diseases are viewed as rather general phenomena inherent in constrained systems and commonly originate from the invertibility condition [6].

The Rarita-Schwinger field coupled to an external electromagnetic field is reduced to a mechanical model with the supplementary condition. By using the constrained mechanical model we trace the origin of the Rarita-Schwinger paradoxes, namely the non-positive definiteness of the anticommutators and the non-causal modes of propagation. The aim of this paper is to show that the Rarita-Schwinger paradoxes have a common origin inherent in the constrained dynamical system.

In § 2 we present the mechanical model with the supplementary condition and analyse the situation corresponding to the occurrence of secondary constraints. In § 3 we show that the Rarita-Schwinger field coupled to an external electromagnetic field is reduced to the constrained mechanical model of § 2 and that the inconsistencies (the Rarita-Schwinger paradoxes) come from the same source, as is expected. One of the spin-$\frac{1}{2}$ components plays the role of a Lagrange multiplier. This corresponds to having a primary constraint arising from the Lagrangian. The final section is devoted to the conclusion. In the appendix the characteristic determinant is obtained from the viewpoint of keeping the manifest covariance in mind.

2. The mechanical model

Consider a system with $N+1$ degrees of freedom described by the Lagrangian

$$L = \mathrm{i}\phi_\mu^\dagger m_{\mu\nu}\dot{\phi}_\nu - \phi_\mu^\dagger a_{\mu\nu}\phi_\nu - \lambda^\dagger c_\nu\phi_\nu - \phi_\mu^\dagger c_\mu^\dagger\lambda - V(\phi_\mu^\dagger, \phi_\nu). \tag{2.1}$$

We assume summation over repeated indices. Here $\phi^{\dagger}_{\mu}$, ϕ_{ν}, (μ, $\nu = 0, 1, 2, \ldots, N$) are the coordinates and the potential term V contains quadratic as well as higher-order terms in the coordinates†. By introducing the Lagrange multipliers $\lambda^{\dagger}$, λ, we have imposed the linear supplementary condition

$$c_{\mu}\phi_{\mu} = 0 \qquad (\phi^{\dagger}_{\mu}c^{\dagger}_{\mu} = 0) \tag{2.2}$$

where the coefficients c_{μ} do not contain the coordinates.

The 'kinetic mass matrix' $m_{\mu\nu}$ and the matrix $a_{\mu\nu}$ are both Hermitian.

Since the matrix m is Hermitian, it can always be diagonalised by a unitary transformation. We assume that this has been done and that

$$m_{\mu\nu} = m_{\mu}\delta_{\mu\nu} \tag{2.3}$$

with no summation over the repeated index of the eigenvalue m_{μ} from now on. This eigenvalue is called the kinetic mass [7].

The kinetic masses are assumed to be non-vanishing but not necessarily all positive. This 'wrong' sign in the kinetic part may create the inconsistencies.

The potential term plays no role in the constraint structure and is therefore dropped from now on. Dropping the potential term V, the Lagrangian can be written

$$L = \mathrm{i}m_{\mu}\phi^{\dagger}_{\mu}\delta_{\mu\nu}\dot{\phi}_{\nu} - \phi^{\dagger}_{\mu}a_{\mu\nu}\phi_{\nu} - \lambda^{\dagger}c_{\nu}\phi_{\nu} - \phi^{\dagger}_{\mu}c^{\dagger}_{\mu}\lambda. \tag{2.4}$$

We now assume that the mass matrix is non-singular in order to analyse in detail the problem inherent in the constrained system. The problem is to show that both the non-positive definiteness of the anticommutators and the non-causal modes of propagation commonly originate from the lack of invertibility of the operator $(dc^{\dagger})$ defined by equation (2.8) below.

The Euler–Lagrange equations corresponding to the Lagrangian (2.4) are

$$\mathrm{i}m_{\mu}\delta_{\mu\nu}\dot{\phi}_{\nu} - a_{\mu\nu}\phi_{\nu} - c^{\dagger}_{\mu}\lambda = 0 \tag{2.5}$$

and

$$c_{\nu}\phi_{\nu} = 0 \tag{2.6}$$

where equation (2.6) implies the supplementary condition.

Taking the time derivative of equation (2.6), we obtain

$$(dc^{\dagger})\lambda + (\mathrm{i}\dot{c}_{\mu}\delta_{\mu\nu} + d_{\mu}a_{\mu\nu})\phi_{\nu} = 0. \tag{2.7}$$

Here we have substituted (2.5) into (2.7) and we have defined $(dc^{\dagger})$ as

$$(dc^{\dagger}) = d_{\mu}c^{\dagger}_{\mu} \tag{2.8}$$

where

$$d_{\mu} = m_{\mu}^{-1}c_{\mu}. \tag{2.9}$$

Assuming the invertibility condition, which is

$$(dc^{\dagger}) \neq 0 \tag{2.10}$$

† The variable ϕ_{μ} could be multicomponent and c_{μ} is in general matrix. However, we suppress extra indices for simplicity. The matrix multiplication rule should therefore be used in the algebra of c_{μ} and its associated quantities. It should also be noted that the quantities with Greek indices do not necessarily imply tensors. See examples in § 3, equations (3.19), (3.20) and (3.23).

we obtain from (2.5) and (2.7) the true equations of motion:

$$\mathrm{i}m_\mu \dot{\phi}_\mu + \mathrm{i}c^\dagger_\mu (dc^\dagger)^{-1}\dot{c}_\nu \phi_\nu - M_{\mu\nu} a_{\nu\lambda}\phi_\lambda = 0. \tag{2.11}$$

Here we have introduced the operator $M_{\mu\nu}$ defined by

$$M_{\mu\nu} = \delta_{\mu\nu} - c^\dagger_\mu (dc^\dagger)^{-1} d_\nu. \tag{2.12}$$

This operator satisfies the following relationships:

$$M_{\mu\lambda}M_{\lambda\nu} = M_{\mu\nu} \tag{2.13}$$

$$d_\mu M_{\mu\nu} = M_{\mu\nu}c^\dagger_\nu = 0. \tag{2.14}$$

Another true equation of motion comes from (2.7) by again taking the time derivative and the result turns out to be

$$\begin{aligned}(dc^\dagger)\dot{\lambda} + [2(\dot{d}c^\dagger) + (d\dot{c}^\dagger) - \mathrm{i}(dad^\dagger)]\lambda + [(\mathrm{i}\ddot{c}_\mu\delta_{\mu\nu} + \dot{d}_\mu a_{\mu\nu} + d_\mu \dot{a}_{\mu\nu})\delta_{\nu\lambda}\\ -\mathrm{i}(\mathrm{i}\dot{c}_\mu\delta_{\mu\nu} + d_\mu a_{\mu\nu})(1/m_\nu)a_{\nu\lambda}]\phi_\lambda = 0\end{aligned} \tag{2.15}$$

where

$$(dad^\dagger) = d_\mu a_{\mu\nu} d^\dagger_\nu. \tag{2.16}$$

Here we have substituted (2.5) into (2.15) in the course of the derivation.

Thus we have obtained the true equations of motion for the coordinates ϕ_μ and the Lagrange multiplier λ [see equations (2.11) and (2.15)).

2.1. Propagation

The principal parts of the true equations of motion (2.11) and (2.15) become

$$\mathrm{i}m_\mu \delta_{\mu\nu}\dot{\phi}_\nu + \ldots = 0 \tag{2.17}$$

$$(dc^\dagger)\dot{\lambda} + \ldots = 0. \tag{2.18}$$

To find the normals n_μ to the characteristic surfaces we replace

$$\partial_\mu \to n_\mu \tag{2.19}$$

in the principal parts of the true equations of motion in covariant form. In the specific frame

$$n_\mu = (n_0, 0, 0, 0) \tag{2.20}$$

the principal parts of the true equations of motion agree, up to a factor, with equations (2.17) and (2.18) after replacement of ∂_0 by n_0. Thus, the characteristic determinant in this frame turns out to be

$$\mathscr{G}(n_0) \propto (n_0)^{N+1}(dc^\dagger)\prod_{\mu=1}^{N} m_\mu. \tag{2.21}$$

2.2 Quantisation

If we assume that

$$c_0 \neq 0 \tag{2.22}$$

equation (2.6) yields

$$\phi_0 = -c_0^{-1}c_k\phi_k \qquad (\phi^\dagger_0 = -\phi^\dagger_i c^\dagger_i c_0^{\dagger -1}) \tag{2.23}$$

where Latin indices from the middle of the alphabet, namely $i, j, k, \ldots$, run from 1 to N. Furthermore, we have

$$\dot{\phi}_0 = -c_0^{-1} c_k \dot{\phi}_k - c_0^{-1} \dot{c}_\nu \phi_\nu. \tag{2.24}$$

To obtain the kinetic energy part of the Lagrangian (2.4) we substitute (2.23) and (2.24) back into it. The result turns out to be

$$L = \mathrm{i}\phi_i^\dagger \mathcal{M}_{ij} \dot{\phi}_j + \ldots \tag{2.25}$$

with

$$\mathcal{M}_{ij} = [\delta_{ij} + c_i^\dagger (d_0 c_0^\dagger)^{-1} d_j] m_j. \tag{2.26}$$

The quantity $\mathcal{M}_{ij}$ is called the 'effective mass' [7].

Let $\pi_j^\dagger$ be the canonical momenta conjugate to the coordinates ϕ_j:

$$\pi_j^\dagger \equiv \partial L / \partial \dot{\phi}_j = \mathrm{i}\phi_i^\dagger \mathcal{M}_{ij}. \tag{2.27}$$

Assuming the usual equal time anticommutators

$$\{\phi_i(x), \pi_k(y)\} = \mathrm{i}\delta_{ik}\delta^{(3)}(x-y) \tag{2.28}$$

we arrive at

$$\{\phi_\mu(x), \phi_\nu^\dagger(y)\} = \delta^{(3)}(x-y) m_\mu^{-1} M_{\mu\nu}. \tag{2.29}$$

Here use has been made of the relationship

$$\mathcal{M}_{ij}(1/m_j) M_{jk} = M_{ij} \mathcal{M}_{jk}(1/m_k) = \delta_{ik} \tag{2.30}$$

as well as (2.23).

We have applied the Hamiltonian formalism proposed by one of the authors [7]. The results we obtained for the system are equivalent to those of Dirac, who studied constrained Hamiltonian systems in general [8]. We underline the simplicity, generality and transparency of the above derivation.

The invertibility condition (2.10), namely

$$(dc^\dagger) \neq 0 \tag{2.31}$$

is responsible for both the propagation and quantisation of the coordinates ϕ_μ. Thus, we have arrived at the conclusion that the anomalies have a common origin inherent in the constrained systems.

3. The Rarita–Schwinger field

The Lagrange density for the Rarita–Schwinger field coupled to an external electromagnetic field can be cast into the form (2.4).

Let us start with the Lagrange density

$$\mathscr{L} = \bar{\psi}_\mu \Lambda^{\mu;\rho}(D) \psi_\rho = \varepsilon^{\mu\rho\sigma\lambda} \bar{\psi}_\mu \gamma_5 \gamma_\sigma D_\lambda \psi_\rho + \mathrm{i} m \bar{\psi}_\mu \sigma^{\mu\rho} \psi_\rho \tag{3.1}$$

where

$$\Lambda^{\mu;\rho}(D) = -[\mathrm{i}(\gamma \cdot D) - m] g^{\mu\rho} + \mathrm{i}(\gamma^\mu D^\rho + D^\mu \gamma^\rho) - \gamma^\mu [\mathrm{i}(\gamma \cdot D) + m] \gamma^\rho. \tag{3.2}$$

Here ψ_μ is the vector-spinor field describing spin-$\frac{3}{2}$ particles with mass m and $\bar{\psi}_\mu$ is the Dirac conjugate to ψ_μ, $\bar{\psi}_\mu = \psi_\mu^\dagger \gamma_0$†. Our metric is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. We have used the convention $\varepsilon_{0123} = 1$.

To introduce coupling to an external electromagnetic field, we have made the replacement

$$\partial_\mu \to D_\mu = \partial_\mu - \mathrm{i}eA_\mu. \tag{3.3}$$

This choice yields

$$[D_\mu, D_\nu] = -\mathrm{i}eF_{\mu\nu} = -\mathrm{i}e(\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{3.4}$$

Next we split the Rarita-Schwinger field ψ_μ into irreducible spin parts according to

$$\phi_i \equiv -P_{i:k}^{(3/2)}\psi_k \tag{3.5}$$

$$\phi_0 \equiv -\gamma_i P_{i:k}^{(1/2)}\psi_k \tag{3.6}$$

$$\lambda \equiv \psi_0 \tag{3.7}$$

where $P_{i:k}^{(a)}$ $(a = \frac{3}{2}, \frac{1}{2})$ is the spin projection operator that projects out the spin-a part from the vector-spinor field and is given by

$$P_{i:k}^{(3/2)} \equiv -P_{ik} = g_{ik} - \tfrac{1}{3}\gamma_i\gamma_k \qquad \text{for } a = \tfrac{3}{2} \tag{3.8}$$

$$P_{i:k}^{(1/2)} = \tfrac{1}{3}\gamma_i\gamma_k \qquad \text{for } a = \tfrac{1}{2} \tag{3.9}$$

with

$$P_{i:k}^{(3/2)} + P_{i:k}^{(1/2)} = I_{i:k} = g_{ik}. \tag{3.10}$$

Thus we have

$$\phi_i = P_{ik}\psi_k = -(g_{ik} - \tfrac{1}{3}\gamma_i\gamma_k)\psi_k \tag{3.11}$$

$$\phi_0 = \gamma_k\psi_k \tag{3.12}$$

$$\lambda = \psi_0. \tag{3.13}$$

The Lagrange density (3.1) can be written in terms of (3.11)-(3.13):

$$\begin{aligned}\mathscr{L} &= \bar{\psi}_\mu\{-[\mathrm{i}(\gamma\cdot D) - m]g^{\mu\rho} + \mathrm{i}(\gamma^\mu D^\rho + D^\mu\gamma^\rho) - \gamma^\mu[\mathrm{i}(\gamma\cdot D) + m]\gamma^\rho\}\psi_\rho \\ &\equiv \mathrm{i}m_\mu\phi_\mu^\dagger P_{\mu\nu}\dot{\phi}_\nu - \phi_\mu^\dagger a_{\mu\nu}\phi_\nu - \lambda^\dagger c_\nu\phi_\nu - \phi_\mu^\dagger c_\mu^\dagger\lambda.\end{aligned} \tag{3.14}$$

† The following representation has been used for the γ matrix:

$$\gamma^0 = \gamma_0 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}$$

$$\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5 = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}$$

$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}$$

with

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma^2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$2\sigma_{\mu\nu} = \mathrm{i}[\gamma_\mu, \gamma_\nu] = \mathrm{i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

$$2g_{\mu\nu} = \{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu.$$

The following notation has been used:

$$\phi_\mu = (\phi_0, \phi_k) \tag{3.15}$$

$$m_\mu = (m_0, m_k) \equiv (-\tfrac{2}{3}, 1) \tag{3.16}$$

$$P_{\mu\nu} = (P_{00}, P_{k0}, P_{0l}, P_{kl}) \equiv (1, 0, 0, -g_{kl} + \tfrac{1}{3}\gamma_k\gamma_l) \tag{3.17}$$

$$a_{\mu\nu} = (a_{00}, a_{k0}, a_{0l}, a_{kl})$$

$$\equiv \{\tfrac{2}{3}[\gamma_0(-\tfrac{1}{3}\mathrm{i}\gamma_D + m) + eA_0], \tfrac{1}{3}\mathrm{i}P_{kj}D_j\gamma_0, -\tfrac{1}{3}\mathrm{i}\gamma_0 D_j P_{jl}, P_{kj}[\gamma_0(\mathrm{i}\gamma_D + m) - eA_0]P_{jl}\} \tag{3.18}$$

$$c_\mu = (c_0, c_k) \equiv [(-\tfrac{2}{3}\mathrm{i}\gamma_D + m), \mathrm{i}D_j P_{jk}] \tag{3.19}$$

$$c_\mu^\dagger = (c_0^\dagger, c_k^\dagger) \equiv [(\tfrac{2}{3}\mathrm{i}\gamma_D + m), \mathrm{i}P_{kj}D_j] \tag{3.20}$$

where

$$\gamma_D \equiv \gamma^l D^l. \tag{3.21}$$

To exhibit the canonical structure of this theory more explicitly we have rewritten the Lagrange density (3.1) in a non-covariant form by splitting the space and time dependence as well as the space and time components. Although the expressions obtained in obtaining the form (3.14) look rather complicated, they exhibit clearly the spin contents of the field as well as their canonical structure.

The operator $(dc^\dagger)$ in (2.8) plays a crucial role for both the propagation and quantisation, as was observed in the previous section. The relationships developed in § 2 can be used in this section by replacing $\delta_{\mu\nu}$ in (2.4) and (2.28) by $P_{\mu\nu}$:

$$\delta_{\mu\nu} \to P_{\mu\nu} \tag{3.22}$$

where $P_{\mu\nu}$ is defined by (3.17). We now have the operator $(dc^\dagger)$ in terms of (3.16)-(3.20):

$$(dc^\dagger) = -\tfrac{3}{2}(m^2 - \tfrac{2}{3}e\boldsymbol{\sigma}\cdot\boldsymbol{H}). \tag{3.23}$$

Note that we have used the relationship

$$\gamma_D^2 + \boldsymbol{D}^2 = -\tfrac{1}{2}e\sigma_{ij}F_{ij} = -e\boldsymbol{\sigma}\cdot\boldsymbol{H}. \tag{3.24}$$

The invertibility condition is not satisfied in (3.23), since the matrix $(dc^\dagger)$ becomes singular on a world sheet, namely

$$\det(dc^\dagger) = (\tfrac{3}{2}m^2)^4[1 - (\tfrac{2}{3}e/m^2)^2\boldsymbol{H}^2]^2. \tag{3.25}$$

This creates all the difficulties so far discussed.

Let $L_{\mu\nu}$ be the resulting coefficient matrix in the principal part of the true equation of motion (A10) in the appendix. This is given by (A15). We find the following relationship in the specific frame $n_\mu = (n_0, 0, 0, 0)$:

$$\mathrm{i}\gamma_0\tfrac{2}{3}(1/m^2)n_0(dc^\dagger) = L_{00}(n_0) \qquad L_{i0}(n_0) = 0. \tag{3.26}$$

Thus we have shown that the operator $(dc^\dagger)$ is responsible for the non-causal modes of propagation.

Quantisation of the Rarita-Schwinger field is easily carried out by the substitution of (3.22) in (2.29). The result is

$$\{\phi_\mu(x), \phi_\nu^\dagger(y)\} = \delta^{(3)}(x-y)m_\mu^{-1}M_{\mu\nu}. \tag{3.27}$$

This is nothing but the Johnson-Sudarshan anticommutator. Here we have again assumed the invertibility condition

$$\det(dc^{\dagger}) \neq 0. \tag{3.28}$$

Thus we conclude that both the non-positive definiteness of the anticommutators and non-causal modes of propagation arise from the same origin, namely the invertibility condition (3.28), inherent in the constrained dynamical systems.

4. Conclusions

Guided by the mechanical model with the linear supplementary condition we have arrived at the conclusion that the Rarita-Schwinger paradoxes, the non-positive definiteness of the anticommutators and the non-causal modes of propagation have the same origin. This is that the lack of invertibility of the operator $(dc^{\dagger})$ is responsible for the difficulties so far discussed.

The same situation has been observed in the constraint mechanical model [9, 10] describing the system with $2(N+1)$ degrees of freedom. We have also discussed the Gribov ambiguity in the non-Abelian gauge field from the invertibility point of view [11].

Acknowledgments

This work was supported in part by the Natural Sciences and Engineering Research Council (NSERC) of Canada. One of us (MK) wishes to thank K Seo and M Kawasaki for checking the 12×12 determinant of (A20) in the special cases.

Appendix. The characteristic determinant

The Euler-Lagrange equation of motion follows from equation (3.1) that

$$\Lambda_{\mu:\rho}(D)\psi^{\rho} = 0. \tag{A1}$$

The primary constraint arises from the 0 component of (A1):

$$\Lambda_{0:\rho}(D)\psi^{\rho} = \gamma_0[\mathrm{i}D_k + (\mathrm{i}\gamma_D - m)\gamma_k]\psi_k = 0. \tag{A2}$$

This implies that ψ_0 plays the role of Lagrange multipliers. The secondary constraints are obtained by contracting (A1) with γ^{μ} and D^{μ}:

$$\{\mathrm{i}D_{\rho} - [\mathrm{i}(\gamma\cdot D) + \tfrac{3}{2}m]\gamma_{\rho}\}\psi^{\rho} = 0 \tag{A3}$$

$$[\mathrm{i}D_{\rho} - \mathrm{i}(\gamma\cdot D)\gamma_{\rho} - \mathrm{i}e(1/m)F_{\rho\alpha}\gamma^{\alpha} - \tfrac{1}{2}e(1/m)\sigma_{\alpha\beta}F^{\alpha\beta}\gamma_{\rho}]\psi^{\rho} = 0. \tag{A4}$$

Solving (A3) and (A4), we obtain

$$(\mathcal{M}\gamma_{\rho} - \mathrm{i}e\kappa F_{\rho\alpha}\gamma^{\alpha})\psi^{\rho} = 0 \tag{A5}$$

where

$$\mathcal{M} \equiv 1 - e\kappa\tfrac{1}{2}\sigma_{\alpha\beta}F^{\alpha\beta} \tag{A6}$$

with

$$\kappa = 2/3m^2. \tag{A7}$$

Solving for $\mathcal{M}\mathrm{i}D_\rho\psi^\rho$ by substituting (A5) back into (A3) we obtain

$$\{\mathcal{M}\mathrm{i}D_\rho-[\mathrm{i}(\gamma\cdot D)+\tfrac{3}{2}m]\mathrm{i}e\kappa F_{\rho\alpha}\gamma^\alpha-R\gamma_\rho\}\psi^\rho=0 \tag{A8}$$

where R is given by

$$R\equiv[\mathcal{M},\mathrm{i}(\gamma\cdot D)]=-e\kappa(D^\alpha F_{\alpha\beta}+F_{\alpha\beta}D^\alpha)\gamma^\beta. \tag{A9}$$

To obtain the true equation of motion, we contract (A1) with $\mathcal{M}$ and substitute (A5) and (A8) into it:

$$\begin{aligned}\mathcal{M}\Lambda_{\mu:\rho}(D)\psi^\rho=\{&-[\mathrm{i}(\gamma\cdot D)-m]\mathcal{M}g_{\mu\rho}-Rg_{\mu\rho}+R_\mu^{(1)}\mathrm{i}D_\rho+R_\mu^{(2)}\gamma_\rho\\&-R_\mu^{(1)}[\mathrm{i}(\gamma\cdot D)+m]\gamma_\rho+(\mathrm{i}D_\mu+\tfrac{1}{2}m\gamma_\mu)\mathrm{i}e\kappa F_{\rho\alpha}\gamma^\alpha\}\psi^\rho=0.\end{aligned} \tag{A10}$$

This is the true equation of motion. Here we have defined the operators $R_\mu^{(i)}$ $(i=1,2)$ as

$$R_\mu^{(1)}\equiv[\mathcal{M},\gamma_\mu]=2\mathrm{i}e\kappa F_{\mu\alpha}\gamma^\alpha \tag{A11}$$

$$\begin{aligned}R_\mu^{(2)}&\equiv[\mathcal{M},\mathrm{i}D_\mu]\\&=-e\kappa\{(D^\alpha F_{\alpha\mu}+F_{\alpha\mu}D^\alpha)-[(\gamma\cdot D)F_{\alpha\mu}\gamma^\alpha+\gamma^\alpha F_{\alpha\mu}(\gamma\cdot D)]\}.\end{aligned} \tag{A12}$$

To find the normals n_μ to the characteristic surfaces we replace

$$\partial_\mu\to n_\mu \tag{A13}$$

in the principal part and calculate the determinant $\mathscr{G}(n)$ of the resulting coefficient matrix $L_{\mu\nu}(n)$. Thus the true equation of motion (A10) yields

$$\mathscr{G}(n)=\det L_{\mu\nu}(n) \tag{A14}$$

where

$$\begin{aligned}L_{\mu\nu}(n)=-\mathrm{i}(\gamma\cdot n)\mathcal{M}g_{\mu\nu}&+e\kappa(n^\alpha F_{\alpha\beta}+F_{\alpha\beta}n^\alpha)\gamma^\beta g_{\mu\nu}\\&-2e\kappa F_{\mu\alpha}\gamma^\alpha n_\nu+2e\kappa F_{\mu\alpha}\gamma^\alpha(\gamma\cdot n)\gamma_\nu-n_\mu e\kappa F_{\nu\alpha}\gamma^\alpha\\&-e\kappa\{(n^\alpha F_{\alpha\mu}+F_{\alpha\mu}n^\alpha)-[(\gamma\cdot n)F_{\alpha\mu}\gamma^\alpha+\gamma^\alpha F_{\alpha\mu}(\gamma\cdot n)]\}\gamma_\nu.\end{aligned} \tag{A15}$$

To avoid excessively cumbersome computations we take the special frame in which

$$n_\mu=(n_0,0,0,0). \tag{A16}$$

This frame plays a crucial role for the true equations of motion in non-covariant form.

Since we have

$$L_{00}(n_0)=\mathrm{i}\gamma_0\kappa n_0(dc^\dagger) \tag{A17}$$

$$L_{i0}(n_0)=0 \tag{A18}$$

the characteristic determinant in the frame (A16) becomes

$$\mathscr{G}(n_0)=\det L_{00}(n_0)\det L_{ij}(n_0) \tag{A19}$$

where $L_{ij}(n_0)$ is a 12×12 matrix and is given by

$$L_{ij}(n_0)=-\mathrm{i}n_0\{e\kappa(-\boldsymbol{\sigma}\cdot\boldsymbol{E}g_{ij}-2E^i\sigma^j)\otimes\rho^2+[(1-e\kappa\boldsymbol{\sigma}\cdot\boldsymbol{H})g_{ij}-2\mathrm{i}e\kappa F_{ik}\sigma^k\sigma^j]\otimes\rho^3\}. \tag{A20}$$

Here the fields $\boldsymbol{H}, \boldsymbol{E}$ have been defined by

$$H^i = \tfrac{1}{2}\varepsilon_{ijk}F_{jk} \qquad (i = 1, 2, 3) \tag{A21}$$

$$E^i = F_{0i} \qquad (i = 1, 2, 3) \tag{A22}$$

with the convention $\varepsilon_{123} = 1$, and $\boldsymbol{\sigma}$ is defined by

$$\sigma^i = \tfrac{1}{2}\varepsilon_{ijk}\sigma_{jk} = \sigma^j \otimes \mathbb{1} \qquad (i = 1, 2, 3). \tag{A23}$$

Thus the characteristic determinant in covariant form turns out to be

$$\begin{aligned}\mathcal{G}(n) = {} & [n^2 + e^2\kappa^2(n \cdot F^{\mathrm{d}})^2]^2\{n^2 + e^2\kappa^2[(n \cdot F^{\mathrm{d}})^2 - (n \cdot F)^2]\}^4 \\ & \times \{n^2 + e^2\kappa^2[9(n \cdot F^{\mathrm{d}})^2 - (n \cdot F)^2]\}^2.\end{aligned} \tag{A24}$$

Here $F^{\mathrm{d}}_{\mu\nu}$ is the dual field of $F_{\mu\nu}$ and is defined by

$$F^{\mathrm{d}}_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{A25}$$

Notice that the invertibility condition for (A5) is

$$\det \mathcal{M} = [1 + e^2\kappa^2(\boldsymbol{E}^2 - \boldsymbol{H}^2)]^2 \neq 0. \tag{A26}$$

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